

ON THE ACCURACY OF THE POST-NEWTONIAN APPROXIMATION*

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We apply standard post-Newtonian methods in general relativity to locate the innermost circular orbit (ICO) of irrotational and corotational binary black-hole systems. We find that the post-Newtonian series converges well when the two masses are comparable. We argue that the result for the ICO which is predicted by the third post-Newtonian (3PN) approximation is likely to be very close to the “exact” solution, within 1% of fractional accuracy or better. The 3PN result is also in remarkable agreement with a numerical calculation of the ICO in the case of two corotating black holes moving on exactly circular orbits. The behaviour of the post-Newtonian series suggests that the gravitational dynamics of two bodies of comparable masses does not resemble that of a test particle on a Schwarzschild background. This leads us to question the validity of some post-Newtonian resummation techniques that are based on the idea that the field generated by two black holes is a deformation of the Schwarzschild space-time.

1 Introduction

The “standard” post-Newtonian approximation, or expansion when the speed of light $c \rightarrow +\infty$, is at the basis of an important body of research, which has provided us in the past with our best picture of the motion of compact objects in general relativity. We quote the pioneering works of Einstein¹, Droste², and DeSitter³, the landmark analysis due to Einstein, Infeld and Hoffmann⁴ of the dynamics of N separated bodies at the first post-Newtonian order (1PN, or $1/c^2$), and the seminal papers by Chandrasekhar and collaborators^{5,6,7} concerning the equations of motion of extended fluid systems, up to the 2.5PN level (at which order appears the first effect due to the reaction to the emission of gravitational radiation). In the case of two compact objects (neutron stars or black holes), we possess the 2.5PN equations of motion of the binary pulsar^{8,9,10,11}, and the 3PN equations of motion of inspiralling compact binaries^{12,13,14,15,16,17,18,19,20,21}. Regarding the problem of the gravita-

*IN “2001: A RELATIVISTIC SPACETIME ODYSSEY”, PROC. OF THE 25TH JOHNS HOPKINS WORKSHOP, EDITED BY IGNACIO CIUFOLINI, DANIELE DOMINICI AND LUCA LUSANNA, WORLD SCIENTIFIC, P. 411 (2001).

tional radiation emitted by inspiralling compact binaries, we have under control most of the gravitational-wave flux up to the 3.5PN order^{22,23,24,25}, including the specific effects of wave tails^{26,27}.

In this paper we focus our attention on the question of the dynamics of black-hole binary systems (henceforth we assume that the compact objects are black holes) and its recent resolution up to the 3PN approximation, corresponding to the formal order $1/c^6$ beyond the Newtonian force law. On the one hand the ADM-Hamiltonian formalism of general relativity has been applied at the 3PN order by Jaranowski and Schäfer^{12,13}, and Damour, Jaranowski and Schäfer^{14,15}. On the other hand a direct 3PN iteration of the equations of motion — instead of a Hamiltonian — in harmonic coordinates (extending the method proposed in Ref.¹¹) has been implemented by Blanchet and Faye^{17,18,19,20}, and de Andrade, Blanchet and Faye²¹. These two independent approaches have succeeded; it has been shown that there exists a unique transformation of the particle's dynamical variables that changes the 3PN harmonic-coordinates Lagrangian²¹ into a Lagrangian whose Legendre transform is exactly identical to the 3PN ADM-coordinates Hamiltonian¹⁴.

In the previous approaches the two black holes are modelled by point particles, solely described by their masses m_1 and m_2 . This is consistent with the very spirit of the post-Newtonian method. The point-particle description has to be supplemented by a process of regularization of the self field of point particles. The standard regularization *à la* Hadamard is at the basis of the works^{12,13,14,15,17,20,21} (an extended version of this regularization has been defined in Refs.^{18,19}). Unfortunately it was shown that at the 3PN order the equations of motion of black holes contain a particular coefficient, i.e. the “static” ambiguity ω_s in the ADM-Hamiltonian formalism^{12,13,14,15} or the parameter λ in the harmonic-coordinates approach^{17,18,19,20,21a}, which cannot be fixed by the Hadamard regularization. The value of this coefficient has been obtained by means of a dimensional regularization instead of the Hadamard one within the ADM-Hamiltonian formalism¹⁶. We shall discuss some implications of this result (i.e. $\omega_s = 0$) regarding the validity of the post-Newtonian expansion.

Let us look at the conserved energy of the black-hole binary in the center-of-mass frame at the 3PN order. Technically this energy follows from the 3PN harmonic-coordinates Lagrangian²¹ or equivalently from the 3PN ADM-coordinates Hamiltonian¹⁴. The energy is conserved only when we neglect the radiation reaction damping at the 2.5PN order. The time derivative of the

^aSee Eq. (8) below for the relation linking together λ and ω_s .

energy is equal to the radiation reaction effect, but for the present discussion we are interested only in the conservative part of the dynamics which is composed of the Newtonian, 1PN, 2PN and 3PN approximations. Specializing to the case of orbits which are circular (apart from the gradual radiation-reaction inspiral)^b yields then the center-of-mass energy E at the 3PN order for circular orbits as a function of the frequency ω of the orbital motion.

Here we want to assess the validity of the post-Newtonian approximation. More precisely we address, and to some extent we answer, the following questions. How accurate is the post-Newtonian expansion for describing the dynamics of binary black hole systems? Is the innermost circular orbit (ICO) of binary black holes, defined by the minimum of the energy function $E(\omega)$, accurately determined at the highest currently known post-Newtonian order? This question is pertinent because the ICO represents a point in the late stage of evolution of the binary which is very relativistic (orbital velocities of the order of 50% of the speed of light). How well does the 3PN approximation as compared with the prediction provided by numerical relativity? What is the validity of the various post-Newtonian resummation techniques^{28,29} which aim at “boosting” the convergence of the standard post-Newtonian approximation?

The previous questions are very interesting but difficult to settle down rigorously. Indeed the very essence of an approximation method is to cope with our ignorance of the higher-order terms in some expansion, but the higher-order terms are precisely the ones which would be needed for a satisfying answer to these problems. So we shall be able to give only some educated guesses and/or plausible answers, that we cannot justify rigorously, but which seem very likely from the standard point of view on the post-Newtonian theory, in particular that the successive orders of approximation get smaller and smaller as they should (in average), with only few accidents occurring at high orders where a particular approximation would be abnormally large with respect to the lower-order ones. Admittedly, in addition, our faith in the estimation we shall give regarding the accuracy of the 3PN order for instance, comes from the historical perspective, thanks to the many successes achieved in the past by the post-Newtonian approximation when confronting the theory and observations (e.g. of the Solar system dynamics and the binary pulsar). It is indeed beyond question, from our past experience, that the post-Newtonian method does work.

There are many other related issues that we shall not address. For in-

^bWe know that most inspiralling compact binaries have circular orbits because of radiation-reaction effects.

stance : is the best presently known post-Newtonian approximation, i.e. 3PN, sufficient for the problem of the coalescence of two black holes in a foreseeable future^c ? A related problem is the qualitative and quantitative influence of the high-order radiation reaction effects which have to be superposed to the conservative part of the dynamics. Finally we shall discuss only the binary's equations of motion, and leave aside the problem of the radiation field. Certainly the accuracy of the 3.5PN gravitational-radiation flux of black-hole binaries which has been computed in Refs.^{24,25} should be discussed in a similar way.

Basically, the point we would like to emphasize^d is that the post-Newtonian approximation, in standard form (without using the resummation techniques advocated in Refs.^{28,29}), is able to locate the ICO of two black holes with an excellent accuracy of the order of 1%, and perhaps much better than that. At first sight this statement is rather surprising, because the dynamics of two black holes at the point of the ICO is so relativistic. Indeed one often hears about the “bad convergence”, or the “fundamental breakdown”, of the post-Newtonian series in the regime of the ICO. However our estimates do show that the 3PN approximation is very good in this regime, and they are also confirmed by the remarkable agreement, we shall detail below, with a numerical calculation byourgoulhon *et al.*^{31,32} of the ICO in the case of two black holes moving on exactly circular orbits (“helical symmetry”). When comparing the post-Newtonian approximation with the numerical simulation we face an interesting problem : since the numerical work has been done for corotating black holes, which spin with the orbital velocity ω , the effects of spins, appropriate to two Kerr black holes, are to be taken into account within our post-Newtonian framework.

Another point we shall develop is that most probably the general-relativistic dynamics of two objects with comparable masses is qualitatively different (in addition of being far more complicated), than the dynamics of a “test” particle with geodesic motion on a fixed background Schwarzschild space-time. Our argument has something to do with the value $\omega_s = 0$ obtained in Ref.¹⁶ for the 3PN regularization ambiguity parameter. Indeed this value, if we believe that it is really the prediction of general relativity, suggests that the two-body interaction is *not* “Schwarzschild-like”, in the sense that it does not seem to admit a light-ring singularity similar to the one of

^cWe have in mind the 3PN approximation considered as an input for the numerical calculation of the black-hole coalescence (assuming that a clear method would exist for implementing the post-Newtonian initial conditions into a numerical scheme).

^dWe are following the recent study in Ref.³⁰.

the Schwarzschild space-time. We then argue that this fact sheds a doubt on the validity, at high post-Newtonian orders, of some resummation techniques (like Padé approximants²⁸) that are based on the assumption that the field generated by two bodies of comparable sizes is a “deformation” of the Schwarzschild metric. *A contrario* we shall find that the value $\omega_s = 0$ gives some convincing evidence that the standard post-Newtonian approximation, based on Taylor rather than Padé approximants, is very accurate.

2 The binding energy of two black-holes

The binding energy, in the center-of-mass frame, is defined as the invariant energy associated with the *conservative* part of the binary’s 3PN dynamics (we ignore the radiation reaction effect at the 2.5PN order). Restricting our consideration to circular orbits, the energy is a function of a single variable, which can be chosen to be the distance r between the two particles in a given coordinate system, or, better, the frequency $\omega = \frac{2\pi}{P}$ of the orbital motion (P is the orbital period). We introduce for convenience the particular frequency-related parameter

$$x \equiv \left(\frac{GM\omega}{c^3} \right)^{2/3}. \quad (1)$$

The mass-energies of the black holes are m_1 and m_2 (they take notably into account the rotational energies); the total mass is $M = m_1 + m_2$. It is important to express the energy in terms of the frequency-related parameter x , instead of some coordinate distance r , because the function $E(x)$ then takes an invariant expression (the same in different coordinate systems).

Having in hands the circular-orbit energy, we then define the innermost circular orbit (ICO) as the minimum, when it exists, of the energy function $E(x)$. The definition is motivated by our comparison with the numerical calculation^{31,32}, because this is precisely that minimum which is computed numerically. In particular, we do not define the ICO as a point of dynamical general-relativistic instability. The circular-orbit energy, developed to the 3PN order, is of the form

$$E(x) = Mc^2 - \frac{\mu c^2 x}{2} \left\{ 1 + a_1(\nu)x + a_2(\nu)x^2 + a_3(\nu)x^3 + \mathcal{O}(x^4) \right\}. \quad (2)$$

The first term is the rest-mass, the next one, proportional to x , is the Newtonian term, and then we have many post-Newtonian correc-

tions, the coefficients of which are known at present only up to the 3PN order^{12,13,14,15,17,18,19,20,21}, and given by

$$a_1(\nu) = -\frac{3}{4} - \frac{\nu}{12}, \quad (3)$$

$$a_2(\nu) = -\frac{27}{8} + \frac{19}{8}\nu - \frac{\nu^2}{24}, \quad (4)$$

$$a_3(\nu) = -\frac{675}{64} + \left[\frac{209323}{4032} - \frac{205}{96}\pi^2 - \frac{110}{9}\lambda \right] \nu - \frac{155}{96}\nu^2 - \frac{35}{5184}\nu^3. \quad (5)$$

We make use of the useful ratio between the reduced and total masses :

$$\nu = \frac{\mu}{M} \quad \text{where} \quad \mu = \frac{m_1 m_2}{M}. \quad (6)$$

This ratio is interesting because of its range of variation,

$$0 < \nu \leq \frac{1}{4}, \quad (7)$$

where $\nu = \frac{1}{4}$ in the equal-mass case and $\nu \rightarrow 0$ in the test-mass limit for one of the bodies. We shall investigate the value of the ICO as predicted by the 3PN energy (2)-(5) in Section 4.

The 3PN coefficient $a_3(\nu)$ involves the regularization-ambiguity parameter λ introduced in Refs.^{17,20} and due to a physical incompleteness in the Hadamard method^{18,19} for regularizing the self-field of point particles. This parameter is equivalent to the parameter ω_s in Refs.^{12,13} and related to it by^{17,15,21} :

$$\lambda = -\frac{3}{11}\omega_s - \frac{1987}{3080}. \quad (8)$$

It has been argued in Ref.³³ that the numerical value of ω_s could be $\simeq -9$, because for such a value some different resummation techniques, when they are implemented at the 3PN order, give approximately the same result for the ICO. Even more, it was suggested³³ that ω_s might be precisely equal to ω_s^* , with

$$\omega_s^* = -\frac{47}{3} + \frac{41}{64}\pi^2 = -9.34 \dots. \quad (9)$$

However, a more recent computation performed with the help of a dimensional regularization instead of the Hadamard regularization, within the ADM-Hamiltonian formalism¹⁶, has yielded

$$\omega_s = 0 \iff \lambda = -\frac{1987}{3080} . \quad (10)$$

Here we adopt $\omega_s = 0$ as the “correct” value predicted by general relativity for the ambiguity parameter. Note that the result (10) is quite different from ω_s^* given by Eq. (9) : this suggests, according to Ref.³³, that different resummation techniques, *viz* Padé approximants²⁸ and effective-one-body methods²⁹, which are designed to “accelerate” the convergence of the post-Newtonian series, do not in fact converge toward the same exact solution (or, at least, not as fast as expected).

The appearance of one ambiguity parameter at the 3PN order is interesting in connection with the so-called effacing principle satisfied by general relativity¹⁰, according to which the equations of motion and the radiation field of gravitationally condensed objects should depend only on their relativistic masses and not on the detailed features of their internal structure — we neglect the tidal effects between the objects. In general relativity this principle follows from the strong equivalence principle (which differs from the Einstein equivalence principle by the inclusion of bodies with self-gravitational interactions and of experiments involving gravitational forces). Indeed, in the freely falling frame of one of the (spherically symmetric) compact objects, we can apply the Birkhoff theorem and find that the external vacuum field depends only on the mass. In consequence we should expect that the parameter λ is a pure number (e.g. a rational fraction), independent of the internal structure of the compact objects.

It would be interesting to confirm Eq. (10) by an independent calculation, hopefully without resort to any regularization scheme. We have in mind a calculation valid for extended (“fluid”) systems, taking *a priori* into account the internal structure of the fluids. Such a calculation would provide, in addition to the determination of λ , an explicit verification of the effacing principle of general relativity. Notice that from the point of view of a calculation valid for extended fluids, it is hard to believe that the ambiguity parameter could depend on π^2 like in Eq. (9).

In the limiting case $\nu \rightarrow 0$, the energy (2)-(5) reduces to the 3PN approximation of the exact expression for a test particle in the Schwarzschild background,

$$E^{\text{Sch}}(x) = \mu c^2 \frac{1 - 2x}{\sqrt{1 - 3x}} . \quad (11)$$

The minimum of that function or ICO occurs at $x_{\text{ICO}}^{\text{Sch}} = \frac{1}{6}$, and we have $E_{\text{ICO}}^{\text{Sch}} = \mu c^2 \left(\sqrt{\frac{8}{9}} - 1 \right)$. We know that the ICO in the case of the Schwarzschild metric is also an innermost *stable* circular orbit (or ISCO), i.e. it corresponds to a point of dynamical instability. Another important feature of Eq. (11) is the singularity at the value $x_{\text{light-ring}}^{\text{Sch}} = \frac{1}{3}$ which corresponds to the famous circular orbit of photons in the Schwarzschild metric (“light-ring” singularity). This orbit can also be viewed as the last *unstable* circular orbit. We can check that the post-Newtonian coefficients $a_n^{\text{Sch}} \equiv a_n(0)$ corresponding to Eq. (11) are given by

$$a_n^{\text{Sch}} = - \frac{3^n (2n - 1)!! (2n - 1)}{2^n (n + 1)!} . \quad (12)$$

They increase with n by roughly a factor 3 at each order. This is simply the consequence of the fact that the radius of convergence of the post-Newtonian series is given by the Schwarzschild light-ring singularity at the value $\frac{1}{3}$. We may therefore recover the light-ring orbit by investigating the limit

$$\lim_{n \rightarrow +\infty} \frac{a_{n-1}^{\text{Sch}}}{a_n^{\text{Sch}}} = \frac{1}{3} = x_{\text{light-ring}}^{\text{Sch}} . \quad (13)$$

3 Accuracy of the post-Newtonian expansion

Let us now discuss the accuracy of the 3PN approximation when estimating the ICO of binary black holes. First of all we make a few order-of-magnitude estimates. At the location of the ICO we shall find (see the next Section) that the frequency-related parameter x defined by Eq. (1) is approximately of the order of 20%. Therefore, we might *a priori* expect that the contribution of the 1PN approximation to the energy at the point of the ICO should be of that order. For the present discussion we take the pessimistic view that the order of magnitude of an approximation represents also the order of magnitude of the higher-order terms which are neglected. We see that the 1PN approximation should yield a rather poor estimate of the “exact” result, but this is quite normal at this very relativistic point where the orbital velocity is $\frac{v}{c} \sim x^{1/2} \sim 50\%$. By the same argument we infer that the 2PN approximation should do

much better, with fractional errors of the order of $x^2 \sim 5\%$, while 3PN will be even better, with the accuracy $x^3 \sim 1\%$.

The simple order-of-magnitude estimate suggests therefore that the 3PN order should be close to the “exact” solution for the ICO to within 1% of fractional accuracy. We think that this is very good, and we should even remember that this estimate is pessimistic, because we can reasonably expect that the neglected higher-order approximations, 4PN and so on, are in fact smaller numerically (e.g. of the order of $x^4 \sim 0.2\%$). But let us keep for the present discussion the 1% guess for the accuracy of the 3PN approximation.

Now the previous estimate makes sense only if the numerical values of the post-Newtonian coefficients in Eqs. (3)-(5) stay roughly of the order of one. If this is not the case, and if the coefficients increase dangerously with the post-Newtonian order n , one sees that the post-Newtonian approximation might in fact be very bad. It has often been emphasized in the literature (see e.g. Refs.^{34,35,28}) that in the test-mass limit $\nu \rightarrow 0$ the post-Newtonian series converges slowly, so the post-Newtonian approximation is not very good in the regime of the ICO. Indeed we have seen that when $\nu = 0$ the radius of convergence of the series is $\frac{1}{3}$ (not so far from $x_{\text{ICO}}^{\text{Sch}} = \frac{1}{6}$), and that accordingly the post-Newtonian coefficients increase by a factor ~ 3 at each order. So it is perfectly correct that in the case of test particles in the Schwarzschild background the post-Newtonian approximation is to be carried out to a high order in order to locate the turning point of the ICO.

Let us immediately remark that this negative conclusion does not matter : indeed we shall never use the post-Newtonian approximation when $\nu \rightarrow 0$ simply because we know the exact result which is given by Eq. (8)^e. Therefore we should not worry about the poor convergence of the post-Newtonian series in the test-mass limit. The post-Newtonian method is useless and even one might say irrelevant when considering the motion of a test particle around a Schwarzschild black hole.

What happens when the two masses are comparable ($\nu = \frac{1}{4}$) ? It is clear that the accuracy of the post-Newtonian approximation depends crucially on how rapidly the post-Newtonian coefficients increase with n . We have seen that in the case of the Schwarzschild metric the latter increase is in turn related to the existence of a light-ring orbit. For continuing the discussion we shall say that the relativistic interaction between two bodies of comparable masses is “Schwarzschild-like” if the post-Newtonian coefficients $a_n(\frac{1}{4})$ increase when $n \rightarrow +\infty$. If this is the case this signals the existence of something like a light-ring singularity which could be interpreted as the deformation, when the mass

^eThe exact result for the radiation field is also known, albeit only numerically.

Table 1. Numerical values of the sequence of coefficients of the post-Newtonian series composing the energy function (2)-(5).

		Newtonian	$a_1(\nu)$	$a_2(\nu)$	$a_3(\nu)$
$\nu = 0$		1	-0.75	-3.37	-10.55
$\nu = \frac{1}{4}$	$\omega_s^* \simeq -9.34$	1	-0.77	-2.78	-8.75
$\nu = \frac{1}{4}$	$\omega_s = 0$	1	-0.77	-2.78	-0.97

ratio ν is “turned on”, of the Schwarzschild light-ring orbit. By analogy with Eq. (13) we can estimate the location of this “pseudo-light-ring” orbit by

$$\frac{a_{n-1}(\nu)}{a_n(\nu)} \sim x_{\text{light-ring}}(\nu) \quad \text{with } n = 3. \quad (14)$$

Here $n = 3$ is the highest known post-Newtonian order. If the two-body problem is “Schwarzschild-like” then the right-hand-side of Eq. (14) is small (say around $\frac{1}{3}$), the post-Newtonian coefficients typically increase with n , and most likely it should be difficult to get a reliable estimate by post-Newtonian methods of the location of the ICO. So we ask : is the gravitational interaction between two comparable masses Schwarzschild-like ?

In Table 1 we present the values of the coefficients $a_n(\nu)$ in the test-mass limit $\nu = 0$ [see Eq. (12) for the analytic expression], and in the equal-mass case $\nu = \frac{1}{4}$ when the ambiguity parameter takes the “uncorrect” value ω_s^* [defined by Eq. (9)] and the correct one $\omega_s = 0$ predicted by general relativity. When $\nu = 0$ we clearly see the expected increase of the coefficients by roughly a factor 3 at each step. Now when $\nu = \frac{1}{4}$ and $\omega_s = \omega_s^*$ we notice that the coefficients increase approximately in the same manner as in the test-mass case $\nu = 0$. This indicates that the gravitational interaction in the case of ω_s^* looks like that in a one-body problem. The associated pseudo-light-ring singularity is estimated using Eq. (14) as

$$x_{\text{light-ring}}(\frac{1}{4}, \omega_s^*) \sim 0.32. \quad (15)$$

The pseudo-light-ring orbit seems to be a very small deformation of the Schwarzschild light-ring orbit given by Eq. (13). In this Schwarzschild-like situation, we should not expect the post-Newtonian series to be very accurate.

Now in the case $\nu = \frac{1}{4}$ but when the ambiguity parameter takes the correct value $\omega_s = 0$, we see that the 3PN coefficient $a_3(\frac{1}{4})$ is of the order of minus one instead of being ~ -10 . This suggests (unless 3PN happens to be quite accidental) that the post-Newtonian coefficients in general relativity

do not increase very much with n . This is an interesting finding because it indicates that the actual two-body interaction in general relativity is *not* Schwarzschild-like. There does not seem to exist something like a light-ring orbit which would be a deformation of the Schwarzschild one. Applying Eq. (14) we obtain as an estimate of the “light-ring” :

$$x_{\text{light-ring}}(\frac{1}{4}, \text{G.R.}) \sim 2.88. \quad (16)$$

It is clear that if we believe the correctness of this estimate we must conclude that there is in fact *no* notion of a light-ring orbit in the real two-body problem. Or, one might say (pictorially speaking) that the light-ring orbit gets hidden inside the horizon of the final black-hole formed by coalescence. Furthermore, if we apply Eq. (14) using the 2PN approximation $n = 2$ instead of the 3PN one $n = 3$, we get the value ~ 0.28 instead of Eq. (16). So at the 2PN order the metric seems to admit a light ring, while at the 3PN order it apparently does not admit any. This erratic behaviour reinforces our idea that it is meaningless (with our present 3PN-based knowledge, and untill fuller information is available) to assume the existence of a light-ring singularity when the masses are equal.

It is impossible of course to be thoroughly confident about the validity of the previous statement because we know only the coefficients up to 3PN order. Any tentative conclusion based on 3PN can be “falsified” when we obtain the next 4PN order. Nevertheless, we feel that the mere fact that $a_3(\frac{1}{4}) = -0.97$ in Table 1 is sufficient to motivate our (tentative) conclusion that the gravitational field generated by two bodies is more complicated than the Schwarzschild space-time. This appraisal should look cogent to relativists and is in accordance with the author’s respectfulness of the complexity of the Einstein field equations.

We want next to comment on a possible implication of our conclusion as regards the so-called post-Newtonian resummation techniques, i.e. Padé approximants^{28,33}, which aim at “accelerating” the convergence of the post-Newtonian series in the pre-coalescence stage, and effective-one-body (EOB) methods^{29,33}, which attempt at describing the late stage of the coalescence of two black holes. These techniques are based on the idea that the gravitational two-body interaction is a “deformation” — with $\nu \leq \frac{1}{4}$ being the deformation parameter — of the Schwarzschild space-time. The Padé approximants are valuable tools for giving accurate representations of functions having some singularities. In the problem at hands they would be justified if the “exact” expression of the energy [whose 3PN expansion is given by Eqs. (2)-(5)] would admit a singularity at some reasonable value of x (e.g. ≤ 0.5). In

the Schwarzschild case, for which Eq. (13) holds, the Padé series converges rapidly²⁸ : the Padé constructed only from the 2PN approximation of the energy — keeping only a_1^{Sch} and a_2^{Sch} — already coincide with the exact result given by Eq. (11). On the other hand, the EOB method maps the post-Newtonian two-body dynamics (at the 2PN or 3PN orders) on the geodesic motion on some effective metric which happens to be a ν -deformation of the Schwarzschild space-time. In the EOB method the effective metric looks like Schwarzschild *by definition*, and we might expect the two-body interaction to own some Schwarzschild-like features.

Our comment is that the validity of these post-Newtonian resummation techniques is questionable because the value $\omega_s = 0$ suggests that the two-body interaction is not Schwarzschild-like. Our doubt is confirmed by the finding of Ref.³³ (already alluded to above) that in the case of the “wrong” ambiguity parameter $\omega_s^* \simeq -9.34$ the Padé approximants and the EOB method at the 3PN order give the same result for the ICO. From the previous discussion we see that this agreement is to be expected because a deformed light-ring singularity seems to exist with ω_s^* . By contrast, in the case of general relativity ($\omega_s = 0$), the Padé and EOB methods give quite different results (*cf.* the figure 2 in Ref.³³). Such a disagreement, we argue, is due to the fact that the assumptions underlying the various resummation techniques are probably not fulfilled, so they may converge toward different solutions. Another confirmation comes from the light-ring singularity which is determined from the Padé approximants at the 2PN order [see Eq. (3.22) in Ref.²⁸] as

$$x_{\text{light-ring}}(\frac{1}{4}, \text{Padé}) \sim 0.44 . \quad (17)$$

This value is rather close to Eq. (15) but strongly disagrees with Eq. (16). Our explanation is that the Padé series converges toward a theory having $\omega_s \simeq \omega_s^*$ and so which is different from general relativity.

Finally we come to the good news that, if really the (absolute value of the) post-Newtonian coefficients when $\nu = \frac{1}{4}$ stay of the order of one as it seems to, this means that the *standard* post-Newtonian approach, based on the standard Taylor approximants, is probably very accurate. The post-Newtonian series is likely to “converge well”, with a “convergence radius” of the order of one^f. Therefore the order-of-magnitude estimates we proposed at the beginning of

^f Actually, the post-Newtonian series could be only asymptotic (hence divergent), but nevertheless it should give excellent results provided that the series is truncated near some optimal order of approximation. In this discussion we assume that the 3PN order is not too far from that optimum.

this Section are probably correct. In particular the 3PN order should be close to the “exact” solution even in the regime of the ICO.

Compare the situation with the case of Schwarzschild, for which we have seen that $x_{\text{ICO}}^{\text{Sch}} = \frac{1}{6}$ is rather close to the value of the convergence radius of the series given by $x_{\text{light-ring}}^{\text{Sch}} = \frac{1}{3}$, hence the poor convergence of the post-Newtonian expansion at the location of the ICO. By contrast, when the two masses have the same size, we have $x_{\text{ICO}}(\frac{1}{4}) \sim 0.2$ (see the Figure 1) which is quite far from the “convergence radius” of the series, that we argued is likely to be of the order of one. We thus expect the post-Newtonian expansion (in the case of two black holes) to be well appropriate near the ICO.

4 Comparison with a numerical simulation

We confront the prediction of the standard (Taylor-based) post-Newtonian approach with a recent result of numerical relativity by Gourgoulhon, Grandclément and Bonazzola^{31,32}. These authors computed numerically the energy of binary black holes under the assumptions of conformal flatness for the spatial metric and of exactly circular orbits. The latter restriction is implemented by requiring the existence of an “helical” Killing vector, time-like inside the light cylinder associated with the circular motion and space-like outside. In the numerical approach^{31,32} there are no gravitational waves, the field is periodic in time, and the gravitational potentials tend to zero at spatial infinity within a restricted model equivalent to solving five out of the ten Einstein field equations. Considering an evolutionary sequence of equilibrium configurations Gourgoulhon *et al.*^{31,32} obtained numerically the circular-orbit energy $E(\omega)$ and looked for the innermost circular orbit or ICO.

The numerical calculation^{31,32} has been performed in the case of *corotating* black holes, which are spinning with the orbital angular velocity ω . For the comparison we must therefore include within our post-Newtonian formalism the effects of spins appropriate to two Kerr black holes rotating at the orbital rate ω . The importance of the effect of spins in corotating systems of neutron stars, for which the ICO is usually determined by the hydrodynamical instability rather than by the effect of general relativity, is well known³⁶. We expect that these effects play some role in the case of black holes as well.

The total relativistic mass of the Kerr black hole is given by⁹

$$m^2 = m_{\text{irr}}^2 + \frac{S^2}{4m_{\text{irr}}^2} , \quad (18)$$

⁹In the formulas (18)-(25) we pose $G = 1 = c$.

where S is the spin, related to the usual Kerr parameter by $S = ma$, and m_{irr} is the irreducible mass given by $m_{\text{irr}} = \frac{\sqrt{A}}{4\pi}$ (A is the hole's surface area). The angular velocity of the corotating black hole is $\omega = \frac{\partial m}{\partial S}$ hence, from Eq. (18),

$$\omega = \frac{S}{2m^3 \left[1 + \sqrt{1 - \frac{S^2}{m^4}} \right]} . \quad (19)$$

Physically this angular velocity is the one of the outgoing photons that remain for ever at the location of the light-like horizon³⁷. Combining Eqs. (18) and (19) we obtain m and S as functions of m_{irr} and ω :

$$m = \frac{m_{\text{irr}}}{\sqrt{1 - 4m_{\text{irr}}^2 \omega^2}} , \quad (20)$$

$$S = \frac{4m_{\text{irr}}^3 \omega}{\sqrt{1 - 4m_{\text{irr}}^2 \omega^2}} . \quad (21)$$

This is the right thing to do since ω is the basic variable describing each equilibrium configuration calculated numerically^{31,32}, and because the irreducible masses are the ones which are held constant along the numerical evolutionary sequences^{31,32}. In the limit of small rotations we have

$$S = I \omega + \mathcal{O}(\omega^3) , \quad (22)$$

where $I = 4m_{\text{irr}}^3$ is the moment of inertia of the Kerr black hole³⁸. Next the total mass-energy is

$$m = m_{\text{irr}} + \frac{1}{2} I \omega^2 + \mathcal{O}(\omega^4) . \quad (23)$$

It involves the standard kinetic energy of the spin.

To take into account all the spin effects our first task is to replace all the masses entering the energy function (2)-(5) by their equivalent expressions in terms of ω and the two irreducible masses. It is clear that the leading contribution is that of the spin kinetic energy given by Eq. (23) and comes from the replacement of the rest mass-energy Mc^2 (where $M = m_1 + m_2$). From Eq. (23) this effect in the case of corotating binaries is of order ω^2 , which means by comparison with Eqs. (1)-(2) that it is equivalent to an "orbital" effect at the 2PN order. Higher-order corrections in Eq. (23), which behave at least like ω^4 , will correspond at least to the orbital 5PN order and are

negligible for the present purpose. In addition there will be a subdominant contribution, of the order of $\omega^{8/3}$ equivalent to 3PN order, which comes from the replacement of the masses into the “Newtonian” part, proportional to $x \propto \omega^{2/3}$, of the energy E [see Eq. (2)]. With the 3PN accuracy we do not need to replace the masses that enter into the post-Newtonian corrections in E , so these masses can be considered to be the irreducible ones in these terms.

Our second task is to include the specific relativistic effects due to the spins, namely the spin-orbit (S.O.) interaction and the spin-spin (S.S.) one. In the case of spins S_1 and S_2 aligned parallel to the orbital angular momentum (and right-handed with respect to the sense of motion) the S.O. energy reads

$$E_{\text{S.O.}} = -\mu (M\omega)^{5/3} \left[\left(\frac{4}{3} \frac{m_1^2}{M^2} + \nu \right) \frac{S_1}{m_1^2} + \left(\frac{4}{3} \frac{m_2^2}{M^2} + \nu \right) \frac{S_2}{m_2^2} \right]. \quad (24)$$

Here we are employing the formula given by Kidder, Will and Wiseman³⁹ who have computed the S.O. contribution as expressed by means of the orbital frequency ω (this is what we need in view of the comparison with the numerical work^{31,32}). The basis of their computation is the work of Barker and O’Connell⁴⁰ who obtained the formula given in terms of the orbital separation r . The derivation of Eq. (24) in Ref.³⁹ takes correctly into account the fact that the relation between the orbital separation r (in a given coordinate system) and the frequency ω depends on the spins. We immediately infer from Eq. (22) that in the case of corotating black-holes the S.O. effect is equivalent to a 3PN orbital effect and thus must be retained with the present accuracy [with this approximation, the masses in Eq. (24) are the irreducible ones]. As for the S.S. interaction (still in the case of spins aligned with the orbital angular momentum) it is given by

$$E_{\text{S.S.}} = \mu \nu (M\omega)^2 \frac{S_1 S_2}{m_1^2 m_2^2}. \quad (25)$$

The S.S. effect can be neglected here because it is of the orbital order 5PN for corotating systems.

Summaryzing, the contributions due to the corotating spins at the 3PN order are three in all : the main one is that of the spin kinetic energy and arises at the 2PN order; then we have two subdominant contributions at the 3PN order coming respectively from a mixing between the spin kinetic energy and the Newtonian orbital energy, and from the S.O. interaction (24). We can neglect all the other terms. Summing up these three contributions we

find that the energy of the corotating spins is (coming back to the notation of Section 2)

$$\Delta E^{\text{corot}}(x) = Mc^2 x \left\{ (2 - 6\nu)x^2 + \left(-\frac{18}{3}\nu + 13\nu^2 \right) x^3 + \mathcal{O}(x^4) \right\}. \quad (26)$$

The complete 3PN energy of the corotating binary is the sum of Eqs. (2)-(5) and (26). Notice that we must now understand all the masses in (2)-(5) and (26) as being the irreducible masses — we no longer indicate the superscripts “irr” —, which for the comparison with the numerical work must be assumed to stay constant when the binary evolves.

The Figure 1 (issued from the work³⁰) presents our results for E_{ICO} in the case of irrotational and corotational binaries. Since ΔE^{corot} , given by Eq. (26), is at least of order 2PN, the result for $1\text{PN}^{\text{corot}}$ is the same as for 1PN in the irrotational case; then, obviously, $2\text{PN}^{\text{corot}}$ takes into account only the leading 2PN corotation effect [i.e. the spin kinetic energy given by Eq. (23)], while $3\text{PN}^{\text{corot}}$ involves also, in particular, the corotational S.O. coupling at the 3PN order. In addition we present in Figure 1 the numerical point obtained by numerical relativity under the assumptions of conformal flatness and of helical symmetry^{31,32}. As we can see the 3PN points, and even the 2PN ones, are rather close to the numerical value. The fact that the 2PN and 3PN values are so close to each other is excellent, and confirms the good accuracy of the approximation we concluded in Section 3. In fact one might say that the role of the 3PN approximation is merely to “prove” the value already given by the 2PN one (but of course, had we not computed the 3PN term, we would not be able to trust very much the 2PN value). As expected, the best agreement we obtain is for the 3PN approximation and in the case of corotation : i.e. the point $3\text{PN}^{\text{corot}}$. However, the 1PN approximation is clearly not precise enough, but this is not surprising in the highly relativistic regime of the ICO.

In conclusion, we find that the location of the ICO computed by numerical relativity, under the helical-symmetry and conformal-flatness approximations, is in good agreement with the post-Newtonian prediction. (See Ref.⁴¹ for the results calculated within the EOB method at the 3PN order, which are close to the ones reported in Figure 1.) This constitutes an appreciable improvement of the previous situation, because we recall that the earlier estimates of the ICO in post-Newtonian theory⁴² and numerical relativity^{43,44} strongly disagreed with each other, and do not match with the present 3PN results (see Ref.³² for further discussion).

Notice in Figure 1 the difference between the energies of the corotational

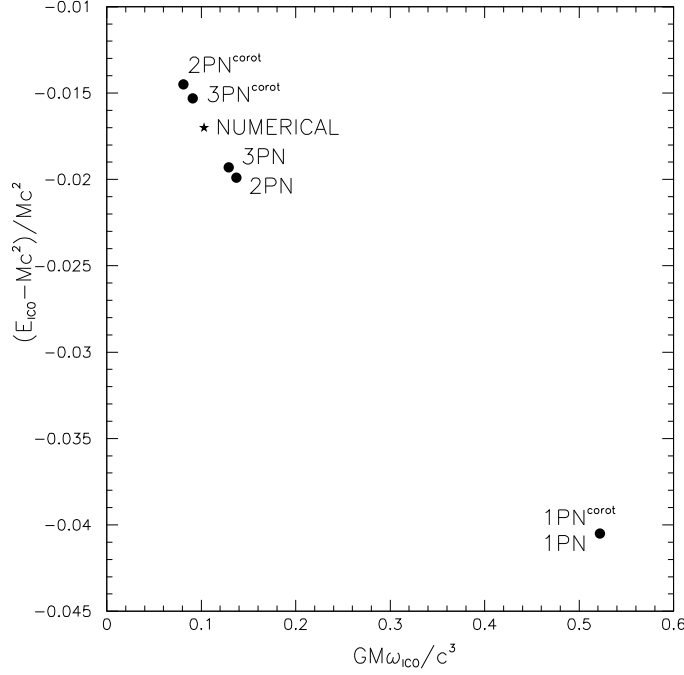


Figure 1. Results for the binding energy $E_{\text{ICO}} - Mc^2$ versus ω_{ICO} in the equal-mass case. The asterisk marks the result calculated by numerical relativity. The points indicated by 1PN, 2PN and 3PN are computed from Eqs. (2)-(5), and correspond to irrotational binaries. The points denoted by 1PN^{corot}, 2PN^{corot} and 3PN^{corot} come from the sum of Eqs. (2)-(5) and (26), and describe corotational binaries. Both 3PN are 3PN^{corot} are shown for $\omega_s = 0$.

and irrotational configurations which amounts to approximately 3.5×10^{-3} units of Mc^2 . We find that this amount is mainly made of the (positive) kinetic energy of the spins, which is about $+5.0 \times 10^{-3}$, but that the S.O interaction at the 3PN order, equal to -1.3×10^{-3} , reduces slightly the effect, while the coupling between spin kinetic and orbital terms is quite negligible, of the order of -0.2×10^{-3} . We feel that these numerical values are physically satisfying, and well in accordance with our order-of-magnitude estimates in Section 3 which showed that in the regime of the ICO the post-Newtonian

series “converges well”, with the successive post-Newtonian approximations becoming numerically smaller and smaller. We have checked that the higher-order spin effects like the S.S. interaction [see Eq. (25)] which arises at the 5PN order for corotating systems are completely negligible numerically (the S.S. effect is about $+0.02 \times 10^{-3}$).

Recently Damour *et al.*⁴¹ computed the effects of spin in corotating black hole binaries within the effective-one-body (EOB) approach. They find a result which differs from our computation³⁰ presented above. Basically they obtain that the kinetic energy of the spins is about $+5.0 \times 10^{-3}$, in agreement with ours, but that the S.O. effect is much larger than in our computation, and nearly compensates in their approach the spin kinetic energy. The net result given in Ref.⁴¹ for the total effect of spins is then about $+1 \times 10^{-3}$, much smaller than our own result. The surprising fact (in the author’s opinion) is that in the EOB method a relativistic effect like the S.O. coupling, which according to the *a priori* expectation based on the estimates of Section 3 should be small, can nearly cancel out a “Newtonian” term, which one would physically consider to be the dominant one. Although Refs.³⁰ and⁴¹ are in very good agreement for the ICO of corotational binaries, and in rather good agreement for irrotational binaries, their discrepancy concerning the numerical value of the S.O. effect remains unexplained. But let us try a guess. As we argued in Section 3 the resummation techniques are probably justified only when the post-Newtonian coefficients in the energy function get bigger and bigger as the order of approximation increases. Our guess would be that perhaps the EOB method is somehow forced to attribute in front of the S.O. term a big post-Newtonian coefficient in order to maintain its internal consistency. With an abnormally big coefficient one might explain the discrepancy.

Acknowledgements

The author is grateful to an anonymous referee of his paper³⁰ for pointing out the interest of looking at the numerical values of the post-Newtonian coefficients.

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